Section 4.3 Implicit Differentiation

Each function we that have found the derivative for this semester have been written in the form:

y = f(x) (or something equivalent with alternate variables) When we are given an equation written in the form

f(x) =

Or

y =

We say that the function has been explicitly written in terms of x.

Sometimes we are asked to find the derivative of functions that are written in more complicated forms. Where it is difficult, if not impossible to express y explicitly in terms of x. This kind of function is called an implicit function.

Here are a few examples for implicit functions. (Notice, none of the equations is solved for y, and it would be difficult to solve any of equations for y.)

- $3x^2 + 6xy = -4y^2 + 8x$
- $\sqrt[3]{y} + 7x = 5y$
- $e^y = 3x 4$

In this section we are going to learn how to differentiate y with respect to x for functions that are not explicitly written in terms of x.

The theory behind the technique needed to find the derivatives in this section involves the chain rule. We can compute the derivatives without completely understanding the theory. I will jump right into the technique we need to find derivatives of implicit functions. The technique is called IMPLICIT DIFFERENTIATION.

- Implicit differentiation is the method do differentiate functions (that is to find $\frac{dy}{dx}$) for implicit functions.
- To compute a derivative using implicit differentiation, we find derivatives normally, except for one difference.
- The difference is when we differentiate a term that has a y we find the derivative normally then multiply by $\frac{dy}{dx}$

Suppose we want to differentiate the implicit function. That is I want to find $\frac{dy}{dx}$ for this function that is not solved for y.

Example: Find $\frac{dy}{dx}$ (I need to use IMPLICIT DIFFERENTIATION.) $y^2 + y = 5x^3 + 7x - 2$

First: Take the derivative of each term with respect to x.

$$\frac{\frac{d}{dx}(y^2) + \frac{d}{dx}(y)}{\frac{d}{dx}(x)} = \frac{d}{dx}(5x^3) + \frac{d}{dx}(7x) - \frac{d}{dx}(2)$$

Next:

Find the derivative of each term that does not have a *y* in the usual manner.

When we find the derivative of a term that has the variable y, find the derivative normally and tack on a $\frac{dy}{dx}$

$$\frac{2y\frac{dy}{dx}}{dx} + \frac{1\frac{dy}{dx}}{dx} = 15x^2 + 7$$

Last: I need to solve for $\frac{dy}{dx}$

I will factor out a common factor of $\frac{dy}{dx}$ on the left side.

$$\frac{dy}{dx}(2y+1) = 15x^2 + 7$$

Now divide both sides by 2y + 1

Answer: $\frac{dy}{dx} = \frac{15x^2 + 7}{2y + 1}$ (this is the desired derivative and is the answer I am searching for)

Example: Find $\frac{dy}{dx}$ (I need to use IMPLICIT DIFFERENTIATION.) $3x^2 + xy = 7$

First: Take the derivative of each term with respect to x.

$$\frac{d}{dx}(3x^2) + \frac{d}{dx}(xy) = \frac{d}{dx}(7)$$

Let us do the easy part first and find the derivative of the $3x^2$ and the 7.

$$6x + \frac{d}{dx}(xy) = 0$$

Now the hard part. Let me find the derivative of the xy in a box to separate this ugliness.

$$\frac{d}{dx}(xy)$$

This needs the product rule:
First factor x Second factor y
Derivative 1 Derivative $1\frac{dy}{dx}$ (I found the derivative of y
and tacked on a dy/dx)
Cross multiply:
 $x * 1\frac{dy}{dx} + 1y$
 $= x\frac{dy}{dx} + y$

 $6x + \frac{d}{dx}(xy) = 0$ $6x + x\frac{dy}{dx} + y = 0$

Subtract 6x and subtract y from each side to get:

$$x\frac{dy}{dx} = -6x - y$$

Now divide by x to get the answer

Answer: $\frac{dy}{dx} = \frac{-6x-y}{x} or - 6 - \frac{y}{x}$

Suppose we want to differentiate in terms of a variable that is not even part of the problem. I can use implicit differentiation to accomplish this.

Example: Use implicit differentiation to determine $\frac{dr}{dt}$.

 $A = 5r^2$ (there is not a *t* in the equation, yet I want to differentiate with respect to *t*.)

First take the derivative of each term with respect to t.

$$\frac{d}{dt}A = \frac{d}{dt}5r^2$$

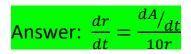
Since neither variable is a t I will tack on a $\frac{d}{dt}$ for each derivative for that variable.

$$1\frac{dA}{dt} = 10r\frac{dr}{dt}$$
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 $\frac{dA}{dt} = 10r\frac{dr}{dt}$

Now divide both sides by 10r

$$\frac{dA/_{dt}}{10r} = \frac{dr}{dt}$$



Here is another example of differentiating in terms of a variable that is not even part of the problem.

Example: Use implicit differentiation to determine $\frac{dr}{dt}$.

V = 3r - 12 (there is not a *t* in the equation, yet I want to differentiate with respect to *t*.)

First take the derivative of each term with respect to *t*.

$$\frac{d}{dt}(V) = \frac{d}{dt}(3r) - \frac{d}{dt}(12)$$

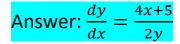
Since neither variable is a t I will tack on a $\frac{d}{dt}$ for each derivative for that variable.

$$1\frac{dV}{dt} = 3\frac{dr}{dt} - 0$$
$$\frac{dV}{dt} = 3\frac{dr}{dt}$$
Divide by 3:
$$\frac{dV/dt}{3} = \frac{dr}{dt}$$
Answer:
$$\frac{dr}{dt} = \frac{dV}{dt}$$

#1-16: Use implicit differentiation to determine $\frac{dy}{dx}$.

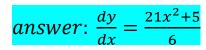
1)
$$y^2 - 3x^2 = 4x - 3$$

2) $y^2 - 2x^2 = 5x - 2$



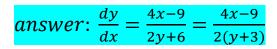
3)
$$5y - 2x^2 = 4x$$

4) $6y - 7x^3 = 5x$



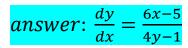
5)
$$y^2 + 3y = 5x^2 + 3x + 1$$

6)
$$y^2 + 6y = 2x^2 - 9x + 1$$



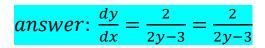
7)
$$3y^2 - y = x^2 - 4x$$

8)
$$2y^2 - y = 3x^2 - 5x$$



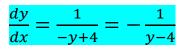
9)
$$y^2 = 6y + x$$

10)
$$y^2 = 3y + 2x$$



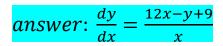
11)
$$3y = y^2 + 4x - 3$$

12)
$$8y = y^2 + 2x - 1$$



13)
$$xy - 3x^2 = 5x$$

14)
$$xy - 6x^2 = 9x$$



115)
$$5xy - 3x^2 = 5x^3$$

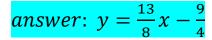
16)
$$2xy - 6x^2 = 7x^3$$

answer:
$$\frac{dy}{dx} = \frac{21x^2 + 12x - 2y}{2x}$$

#17-20: Find the equation of the line tangent to the graph at the indicated point. (Hint, these derivatives have been calculated above.)

17) $y^2 - 3x^2 = 4x - 3$; (1,2)

18) $y^2 - 2x^2 = 5x - 2$; (2,4) this was computed in problem 2: $\frac{dy}{dx} = \frac{4x+5}{2y}$

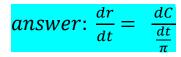


19)
$$xy - 3x^2 = 5x$$
; (2, -1)

20) $xy - 6x^2 = 9x$; (1,3) This was computed in problem 14: $\frac{dy}{dx} = \frac{12x - y + 9}{x}$ #21-28: Use implicit differentiation to determine $\frac{dr}{dt}$.

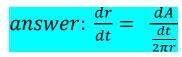
21) $C = 2\pi r$

22) $C = \pi r$



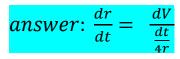
23)
$$A = 5r^2$$

24) $A = \pi r^2$



25)
$$V = 5 + 6r^2$$

26)
$$V = 2r^2 + 8$$



27)
$$V = \frac{2}{3}\pi r^3$$

28)
$$V = \frac{4}{3}\pi r^3$$

