Section 4.3 Implicit Differentiation

Each function we that have found the derivative for this semester have been written in the form:

$$y = f(x) \ \ (or\ something\ equivalent\ with\ alternate\ variables)$$

When we are given an equation written in the form

$$f(x) =$$

Or

$$y =$$

We say that the function has been explicitly written in terms of $x$.

Sometimes we are asked to find the derivative of functions that are written in more complicated forms. Where it is difficult, if not impossible to express $y$ explicitly in terms of $x$. This kind of function is called an implicit function.

Here are a few examples for implicit functions. (Notice, none of the equations is solved for $y$, and it would be difficult to solve any of equations for $y$.)

- $3x^2 + 6xy = -4y^2 + 8x$
- $\sqrt[3]{y} + 7x = 5y$
- $e^y = 3x - 4$

In this section we are going to learn how to differentiate $y$ with respect to $x$ for functions that are not explicitly written in terms of $x$.

The theory behind the technique needed to find the derivatives in this section involves the chain rule. We can compute the derivatives without completely understanding the theory. I will jump right into the technique we need to find derivatives of implicit functions. The technique is called IMPLICIT DIFFERENTIATION.

- Implicit differentiation is the method do differentiate functions (that is to find $\frac{dy}{dx}$ ) for implicit functions.

- To compute a derivative using implicit differentiation, we find derivatives normally, except for one difference.

- The difference is when we differentiate a term that has a $y$ we find the derivative normally then multiply by $\frac{dy}{dx}$

Suppose we want to differentiate the implicit function. That is I want to find $\frac{dy}{dx}$ for this function that is not solved for $y$.

Example: Find $\frac{dy}{dx}$ ( I need to use IMPLICIT DIFFERENTIATION.)

$y^2 + y = 5x^3 + 7x - 2$

First: Take the derivative of each term with respect to $x$.

$\frac{d}{dx}(y^2) + \frac{d}{dx}(y) = \frac{d}{dx}(5x^3) + \frac{d}{dx}(7x) - \frac{d}{dx}(2)$

Next:

Find the derivative of each term that does not have a $y$ in the usual manner.

When we find the derivative of a term that has the variable $y$, find the derivative normally and tack on a $\frac{dy}{dx}$

$2y\frac{dy}{dx} + 1\frac{dy}{dx} = 15x^2 + 7$

Last: I need to solve for $\frac{dy}{dx}$

I will factor out a common factor of $\frac{dy}{dx}$ on the left side.

$$\frac{dy}{dx}(2y + 1) = 15x^2 + 7$$

Now divide both sides by $2y + 1$

$Answer$: $\frac{dy}{dx} = \frac{15x^2+7}{2y+1}$ (this is the desired derivative and is the answer I am searching for)

Example: Find $\frac{dy}{dx}$ ( I need to use IMPLICIT DIFFERENTIATION.)

$3x^2 + xy = 7$

First: Take the derivative of each term with respect to $x$.

$\frac{d}{dx}(3x^2) + \frac{d}{dx}(xy) = \frac{d}{dx}(7)$

Let us do the easy part first and find the derivative of the $3x^2 and\ the\ 7.$

$6x + \frac{d}{dx}(xy) = 0$

Now the hard part. Let me find the derivative of the $xy$ in a box to separate this ugliness.

$$\frac{d}{dx}(xy)$$

This needs the product rule:

First factor $x$ Second factor $y$

Derivative $1$ Derivative $1\frac{dy}{dx}$ (I found the derivative of y and tacked on a $dy/dx$)

Cross multiply:

$x * 1\frac{dy}{dx} + 1y$

$= x\frac{dy}{dx} + y$

$6x + \frac{d}{dx}(xy) = 0$

$6x + x\frac{dy}{dx} + y = 0$

Subtract $6x$ and subtract $y$ from each side to get:

$x\frac{dy}{dx} = -6x - y$

Now divide by $x$ to get the answer

Answer: $\frac{dy}{dx} = \frac{-6x-y}{x}$ $or$ $-6 - \frac{y}{x}$

Suppose we want to differentiate in terms of a variable that is not even part of the problem. I can use implicit differentiation to accomplish this.

Example: Use implicit differentiation to determine $\frac{dr}{dt}$.

$A = 5r^2$ (there is not a $t$ in the equation, yet I want to differentiate with respect to $t$.)

First take the derivative of each term with respect to $t$.

$$\frac{d}{dt}A = \frac{d}{dt}5r^2$$

Since neither variable is a $t$ I will tack on a $\frac{d}{dt}$ for each derivative for that variable.

$$1\frac{dA}{dt} = 10r\frac{dr}{dt}$$

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Now divide both sides by $10r$

$$\frac{dA/dt}{10r} = \frac{dr}{dt}$$

Answer: $\frac{dr}{dt} = \frac{dA/dt}{10r}$

Here is another example of differentiating in terms of a variable that is not even part of the problem.

Example: Use implicit differentiation to determine $\frac{dr}{dt}$.

$V = 3r - 12$ (there is not a $t$ in the equation, yet I want to differentiate with respect to $t$.)

First take the derivative of each term with respect to $t$.

$$\frac{d}{dt}(V) = \frac{d}{dt}(3r) - \frac{d}{dt}(12)$$

Since neither variable is a $t$ I will tack on a $\frac{d}{dt}$ for each derivative for that variable.

$$1\frac{dV}{dt} = 3\frac{dr}{dt} - 0$$

$$\frac{dV}{dt} = 3\frac{dr}{dt}$$

Divide by 3:

$$\frac{dV/dt}{3} = \frac{dr}{dt}$$

Answer: $\frac{dr}{dt} = \frac{dV/dt}{3}$

#1-16: Use implicit differentiation to determine $\frac{dy}{dx}$.

1) $y^2 - 3x^2 = 4x - 3$

2) $y^2 - 2x^2 = 5x - 2$

Answer: $\frac{dy}{dx} = \frac{4x+5}{2y}$

3) $5y - 2x^2 = 4x$

4) $6y - 7x^3 = 5x$

$answer: \frac{dy}{dx} = \frac{21x^2+5}{6}$

5) $y^2 + 3y = 5x^2 + 3x + 1$

6) $y^2 + 6y = 2x^2 - 9x + 1$

$answer: \frac{dy}{dx} = \frac{4x-9}{2y+6} = \frac{4x-9}{2(y+3)}$

7) $3y^2 - y = x^2 - 4x$

8) $2y^2 - y = 3x^2 - 5x$

$answer: \frac{dy}{dx} = \frac{6x-5}{4y-1}$

9) $y^2 = 6y + x$

10) $y^2 = 3y + 2x$

$answer: \frac{dy}{dx} = \frac{2}{2y-3} = \frac{2}{2y-3}$

11) $3y = y^2 + 4x - 3$

12) $8y = y^2 + 2x - 1$

$$\frac{dy}{dx} = \frac{1}{-y+4} = -\frac{1}{y-4}$$

13) $xy - 3x^2 = 5x$

14) $xy - 6x^2 = 9x$

$answer: \frac{dy}{dx} = \frac{12x - y + 9}{x}$

115) $5xy - 3x^2 = 5x^3$

16) $2xy - 6x^2 = 7x^3$

$answer$: $\frac{dy}{dx} = \frac{21x^2+12x-2y}{2x}$

#17- 20: Find the equation of the line tangent to the graph at the indicated point. (Hint, these derivatives have been calculated above.)

17) $y^2 - 3x^2 = 4x - 3; (1,2)$

18) $y^2 - 2x^2 = 5x - 2;\ (2,4)$

$this\ was\ computed\ in\ problem\ 2: \frac{dy}{dx} = \frac{4x+5}{2y}$

$answer:\ y = \frac{13}{8}x - \frac{9}{4}$

19) $xy - 3x^2 = 5x; \;\; (2, -1)$

20) $xy - 6x^2 = 9x; \;\; (1,3)$ This was computed in problem 14:

$\frac{dy}{dx} = \frac{12x - y + 9}{x}$

$answer: \; y = 18x - 15$

#21-28: Use implicit differentiation to determine $\frac{dr}{dt}$.

21) $C = 2\pi r$

22) $C = \pi r$

$answer: \frac{dr}{dt} = \frac{dC}{\frac{dt}{\pi}}$

23) $A = 5r^2$

24) $A = \pi r^2$

$answer: \frac{dr}{dt} = \frac{\frac{dA}{dt}}{2\pi r}$

25) $V = 5 + 6r^2$

26) $V = 2r^2 + 8$

$answer: \frac{dr}{dt} = \frac{\frac{dV}{dt}}{4r}$

27) $V = \frac{2}{3}\pi r^3$

28) $V = \frac{4}{3}\pi r^3$

$answer: \frac{dr}{dt} = \frac{dV}{\frac{dt}{4\pi r^2}}$